Manifestly covariant formalism for electromagnetism in chiral media

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A formalism of electromagnetism covariant under the complex rotation group is first described and then discussed thoroughly, in the case of two-dimensional electromagnetic fields, in homogeneous, isotropic chiral media. As a result, we obtain the generalized Descartes-Snell laws, the Fresnel relations for reflection and refraction, as well as the Brewster condition for plane waves incident on a surface of discontinuity between chiral and achiral media.

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I. INTRODUCTION

The Maxwell-Heaviside equations covariant under the full Lorentz group L are not manifestly covariant because, on the one hand, the curl and the time derivative operator do not form a four-vector and since, on the other hand, the electric and magnetic fields $\mathbf{E},\mathbf{H}$ are the components $F_{i,j}$ and $F_{0,l}$ of the electromagnetic field tensor $E_{\mu\nu}$ $(i,j,k,=1,2,3;\mu,\nu=0,1,2,3)$.

Using a manifestly covariant formalism makes calculations easier and safer. Now in a chiral medium, electromagnetism is not covariant under the full Lorentz group but only under its connected component L_0 which does not include inversions with respect to space and time axes. An electromagnetic formalism covariant under the complex rotation group O $(3,\mathbb{C})$, isomorphic to L_0, were developed many years ago [1,2] for homogeneous isotropic achiral media. Here I extend this formalism to any homogeneous media and I prove that it is particularly well suited to chiral media.

The plan of this paper is as follows. I first describe the complex three-dimensional electromagnetism and the corresponding Fourier transform. Then, the formalism is discussed thoroughly for homogeneous chiral media in the case where the electromagnetic fields do not depend on one coordinate. As an application, I consider the propagation of plane waves in homogeneous isotropic chiral and achiral media when there exists a surface of discontinuity.

Note that many people name bi-isotropic what we name chiral.

II. THREE-DIMENSIONAL COMPLEX FORMALISM

I start with a discussion of the constitutive relations. According to Post [3] the covariant constitutive relations are

$$\mathbf{D}=\eta\mathbf{E}+\gamma\mathbf{B}\ ,\quad \mathbf{H}=\gamma^{\dagger}\mathbf{E}+\chi\mathbf{B}\ , \tag{1}$$

where $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{D}$ the electric displacement, and $\mathbf{B}$ the magnetic induction. The matrices η, γ, χ satisfy the relations

$$\eta=\eta^{\dagger}\ ,\quad \chi=\chi^{\dagger}\ ,\quad \mathrm{Tr}\gamma=0\ , \tag{2}$$

where the symbol $\dagger$ denotes the Hermitian conjugation. It is more convenient to define the constitutive relations in the Tellegen representation [4],

$$\mathbf{D}=\epsilon\mathbf{E}+\alpha\mathbf{H}\ ,\quad \mathbf{B}=\mu\mathbf{H}+\beta\mathbf{E}\ , \tag{3}$$

with, according to (1),

$$\epsilon=\eta-\gamma\chi^{-1}\gamma^{\dagger}\ ,\quad \alpha=\gamma\chi^{-1}\ ,\quad \mu=\chi^{-1}\ ,\quad \beta=-\chi^{-1}\gamma^{\dagger}\ . \tag{4}$$

ϵ, α, μ, β are real constants in homogeneous media as considered here. Let $F_{\mu\nu}$ and $G_{\mu\nu}$ $(\mu,\nu=0,1,2,3)$ be the antisymmetric field tensors with components (E,H) and (D,B), respectively. Each self-dual tensor (we use the summation convention),

$$\tilde{\mathbf{F}}^{\pm}_{\mu\nu}=F_{\mu\nu}\pm\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\ ,\quad \tilde{G}^{\pm}_{\mu\nu}=G_{\mu\nu}\pm\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}\ ,\quad i=\sqrt{-1}\ , \tag{5}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the four-dimensional permutation tensor, has three independent complex components defining a complex vector

$$\tilde{\mathbf{F}}^{\pm}_j=E_j\pm iH_j\ ,\quad \tilde{G}^{\pm}_j=D_j\pm iB_j\ . \tag{6}$$

Then using the vectors (6), the Maxwell-Heaviside equations

$$\nabla\times\mathbf{E}=-\partial_{x_0}\mathbf{B}\ ,\quad \nabla\times\mathbf{H}=\partial_{x_0}\mathbf{D}\ ,\quad x_0=ct\ , \tag{7}$$

$$\nabla\cdot\mathbf{B}=0\ ,\quad \nabla\cdot\mathbf{D}=0\ , \tag{7'}$$

become

$$\nabla\times\tilde{\mathbf{F}}^{\pm}=\pm i\partial_{x_0}\mathbf{G}^{\pm}\ ,\quad \nabla\cdot\tilde{\mathbf{G}}^{\pm}=0\ . \tag{8}$$

These equations simplify in isotropic media, where $\epsilon,\mu,\alpha,\beta$ are scalar because the four-vectors $\tilde{\mathbf{F}}^{\pm},\tilde{\mathbf{G}}^{\pm}$ may be reduced to only two vectors. Leaving the case of anisotropic media for a forthcoming paper, I present now the isotropic formalism.

Let us consider the two complex vectors, where $p^{\pm},q^{\pm}$ are complex scalars to be determined later,

$$\mathbf{\Lambda}^{\pm}=p^{\pm}\mathbf{E}\pm iq^{\pm}\mathbf{H}\ . \tag{9}$$

Then I may write the Maxwell equations (8)

$$\nabla\times\Lambda^{\pm}=\pm i\eta^{\pm}\partial_{x_0}\Lambda^{\pm}\ ,\quad \nabla\cdot\Lambda^{\pm}=0\ , \tag{10}$$

n^+ and n^- are complex refractive indices and one sees at once from (10) that the fields $\Lambda^{\pm}$ are solutions of the wave equation

$$\Delta\Lambda^{\pm}-(n^{\pm})^2\partial_{x_0}^2\Lambda^{\pm}=0\ , \tag{10'}$$

where Δ is the Laplacian operator. Then taking into account (3), (7), and (10) I get the relations

$$\begin{aligned} n^{\pm}p^{\pm}&=q^{\pm}\epsilon\pm ip^{\pm}\beta\ ,\\ n^{\pm}q^{\pm}&=p^{\pm}\mu\mp iq^{\pm}\alpha\ , \end{aligned} \tag{11}$$

which determines $p^{\pm},q^{\pm}$ when the system (9) has a solution. This happens if the following condition is fulfilled:

$$(n^{\pm}\mp i\beta)(n^{\pm}\pm i\alpha)=\epsilon\mu\ . \tag{11'}$$

For an achiral medium $\alpha=\beta=0$, one has the solution $n^{\pm}=(\epsilon\mu)^{1/2}$, $p^{\pm}=\epsilon^{1/2}$, $q^{\pm}=\mu^{1/2}$, leading to $\Lambda^{\pm}=\epsilon^{1/2}\mathbf{E}\pm i\mu^{1/2}\mathbf{H}$, used in the usual complex formalism [1,2] and recently discussed in the spinor form [5].

III. COVARIANT FOURIER TRANSFORM AND WAVE PACKET

In this section, I only use Λ^+ that I rename Λ since it is easy to transpose the results of Λ^+ to Λ^-. I also write n for n^+.

Let us now consider a wave packet propagating in a medium with the constitutive relations (3), (4). Since the plane wave

$$e^{i(k_0x_0-nk_jx_j)} \tag{12}$$

with $k_0^2=|k|^2$ is a solution of the wave equation (10'), I represent the wave packet by its covariant Fourier transform

$$\Lambda_j(x_l,x_0)=\frac{1}{(2\pi)^{3/2}}\int e^{i(k_0x_0-nk_jx_j)}\delta(k_0^2-|k|^2)\times\hat{\phi}_j(k_l,k_0)dk_0dk_l\ , \tag{13}$$

where δ is the Dirac distribution and $|k|^2=k_1^2+k_2^2+k_3^2$. The indices take the values 1,2,3 and we use the summation convention. The Dirac distribution means that, in fact, the integration is not carried out over the whole three-dimensional space time but only on the two cones $k_0=\pm|k|$. These cones are separately Lorentz invariant so that we arrive at the following Lorentz-invariant decomposition

$$\Lambda_j(x_l,x_0)=\Lambda_{j,+}(x_l,x_0)+\Lambda_{j,-}(x_l,x_0)\ , \tag{14}$$

with

$$\Lambda_{j,\pm}(x_l,x_0)=\frac{1}{(2\pi)^{3/2}}\int e^{\pm i(k_0x_0-nk_lx_l)}\delta(k_0^2-|k|^2)\times\hat{\phi}_{j,\pm}(k_l,k_0)dk_0dk_l\ , \tag{15}$$

and

$$\hat{\phi}_{j,\pm}(k_l,k_0)=U(k_0)\hat{\phi}_j(\pm k_l,\pm k_0)\ , \tag{15'}$$

where U is the Heaviside function.

Carrying out in (15) the integration on k_0 gives, with $k_0=+|k|$,

$$\Lambda_{j,\pm}(x_l,x_0)=\frac{1}{(2\pi)^{3/2}}\int\frac{1}{2k_0}e^{\pm i(k_0x_0-nk_lx_l)}\times\hat{\phi}_{j,\pm}(k_l,k_0)dk_l\ . \tag{16}$$

Then, I use the normalization

$$\hat{\phi}_{j,\pm}(k_l,k_0)=\frac{1}{(2k_0)^{1/2}}\phi_{j,\pm}(k_l)\ , \tag{17}$$

and from now on, I only consider wave packets made of the time-harmonic processes $e^{+i|k|x_0}$ that I rename $e^{i|k|x_0}$. I also write Λ_j, ϕ_j, for $\Lambda_{j,+}$, $\phi_{j,+}$. Finally I get

$$\Lambda_j(x_l,x_0)=\frac{1}{(2\pi)^{3/2}}\int\frac{1}{(2k_0)^{1/2}}e^{i(k_0x_0-nk_lx_l)}\times\phi_j(k_l)dk_l\ . \tag{18}$$

Using the polar coordinates

$$k_l=|k|c_1,\quad k_2=|k|c_2,\quad k_3=|k|c_3\ , \tag{19}$$

with

$$c_1=\cos\theta\cos\varphi\ ,\quad c_2=\sin\varphi\ ,\quad c_3=\sin\theta\cos\varphi\ , \tag{19'}$$

the relation (18) becomes

$$\Lambda_j(x_l,x_0)=\frac{1}{(2\pi)^{3/2}}\int_0^{\infty}\tfrac{1}{2}|k|^{1/2}d|k|\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\,e^{i|k|(x_0-nc_lx_l)}\phi_j(|k|,\theta,\varphi)\ , \tag{20}$$

and for a simple time-harmonic process one has

$$\phi_j(|k|,\theta,\varphi)=\delta(|k|-k_0)\psi_j(\theta,\varphi)\ , \tag{21}$$

so that I get from (20)

$$\Lambda_j(x_l,x_0)=\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\,e^{-ik_0nc_lx_l}\psi_j(\theta,\varphi)\ . \tag{22}$$

In the next section I discuss the formalism of Secs. II and III when the electromagnetic field does not depend on one coordinate.

IV. TWO-DIMENSIONAL ELECTROMAGNETIC FIELD

A. Equations

A problem which is completely independent of one Cartesian coordinate, say y, is said to be two dimensional (some people prefer to call it quasi two dimensional). The important point is that the problems of this type are essentially of scalar nature [6]. For such a problem the Maxwell equations (10) reduce to

$$n\partial_{x_0}\Lambda_x(x_\nu,x_0)=i\partial_z\Lambda_y(x_\nu,x_0)\ ,$$
$$n\partial_{x_0}\Lambda_y(x_\nu,x_0)=i[\partial_x\Lambda_z(x_\nu,x_0)-\partial_z\Lambda_x(x_\nu,x_0)]\ , \qquad (23)$$
$$n\partial_{x_0}\Lambda_z(x_\nu,x_0)=-i\partial_x\Lambda_y(x_\nu,x_0)\ .$$

From now on, the Greek letter indices take the values 1,3 corresponding to the x,z coordinates and we remind the reader that $\mathbf{\Lambda}$ and n are identified, respectively, with $\mathbf{\Lambda}^+$ and n^+.

The substitution of (18) into (23) gives the algebraic system of equations

$$ik_0n\phi_x(k_\nu)-k_3n\phi_y(k_\nu)=0\ ,$$
$$ik_0n\phi_y(k_\nu)+k_3n\phi_x(k_\nu)-k_1n\phi_z(k_\nu)=0\ , \qquad (23')$$
$$ik_0n\phi_z(k_\nu)+k_1n\phi_y(k_\nu)=0\ .$$

This homogeneous system has a solution only if the determinant of its coefficients is zero; that is, of course, if

$$k_0^2=k_1^2+k_3^2\ . \qquad (24)$$

Then, the system (23′) supplies the components ϕ_ν in terms of ϕ_y and using the polar coordinates (19′) with $\varphi=0$ I get explicitly, for the time-harmonic process (21),

$$\psi_\nu(\theta)=F_\nu(\theta)\psi_y(\theta)\ , \qquad (25)$$

with

$$F_1(\theta)=-i\sin\theta\ ,\quad F_3(\theta)=i\cos\theta\ . \qquad (25')$$

Then substituting (25′) into (22) and restoring the distinction between $\mathbf{\Lambda}^+$, n^+ and $\mathbf{\Lambda}^-$, n^-, I get

$$\Lambda_\nu^\pm(x_\nu,x_0)=\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}\int_0^{2\pi}e^{-ik_0n^\pm c_\mu x^\mu}\times F_\nu^\pm(\theta)\psi^\pm(\theta)d\theta\ , \qquad (26a)$$

$$\Lambda_y^\pm(x_\nu,x_0)=\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}\int_0^{2\pi}e^{-ik_0n^\pm c_\mu x_\mu}\psi^\pm(\theta)d\theta\ , \qquad (26b)$$

where $\psi^+(\theta)$ and $\psi^-(\theta)$ are some arbitrary complex functions.

B. Boundary conditions

Let us consider a time-harmonic wave packet propagating in a medium made of two materials with different constitutive relations of the Tellegen type (3),(4) and with an interface in the plane $x=0$. I use the relations (26) in the *second* medium and the same relations with letters with a prime in the *first* medium containing the incident wave (this convention simplifies the writing somewhat).

Let m denote the normal to the interface S. Then, in terms of the $\mathbf{E},\mathbf{H},\mathbf{B},\mathbf{D}$ fields, the boundary conditions on S are [7]

$$\epsilon_{jkl}m_k(E_l-E_l')=0\ ,\quad \epsilon_{jkl}m_k(H_l-H_l')=0\ , \qquad (27a)$$
$$m_j(D_j-D_j')=0\ ,\quad m_j(B_j-B_j')=0\ . \qquad (27b)$$

ϵ_{jkl} is the permutation tensor and we use the summation convention. Now from (9) one deduces easily

$$\mathbf{E}+i\mathbf{H}=(\gamma_1^++\gamma_2^+)\mathbf{\Lambda}^++(\gamma_1^--\gamma_2^-)\Lambda^-\ , \qquad (28)$$

and from (7) and (10)

$$\mathbf{D}+i\mathbf{B}=n^+(\gamma_1^++\gamma_2^+)\Lambda^+-n^-(\gamma_1^--\gamma_2^-)\Lambda^-\ , \qquad (28')$$

with

$$\gamma_1^\pm=[p^\pm+q^\pm(q^\mp)^{-1}p^\mp]^{-1}\ ,$$
$$\gamma_2^\pm=[q^\pm+p^\pm(p^\mp)^{-1}q^\mp]^{-1}\ , \qquad (29)$$

so that the boundary conditions (27) become

$$\epsilon_{jkl}m_k[(\gamma_1^++\gamma_2^+)\mathbf{\Lambda}^++(\gamma_1^--\gamma_2^-)\Lambda^-]_l=\epsilon_{jkl}m_k[\gamma_1^++\gamma_2^+)\Lambda^++(\gamma_1^--\gamma_2^-)\mathbf{\Lambda}^-]_l'\ , \qquad (30a)$$

$$m_j[n^+(\gamma_1^++\gamma_2^+)\mathbf{\Lambda}^+-n^-(\gamma_1^--\gamma_2^-)\mathbf{\Lambda}^-]_j=m_j[n^+(\gamma_1^++\gamma_2^+)\mathbf{\Lambda}^+-n^-(\gamma_1^--\gamma_2^-)\mathbf{\Lambda}^-]_j'\ . \qquad (30b)$$

Since I assume the interface in the plane $x=0$, one has $m_j=\delta_{j1}$ where δ_{jk} is the Kronecker symbol and the boundary conditions (30) reduce to

$$[(\gamma_1^++\gamma_2)^+\mathbf{\Lambda}^++(\gamma_1^--\gamma_2^-)\Lambda^-]_{y,z}=[(\gamma_1^++\gamma_2)^+\mathbf{\Lambda}^++(\gamma_1^--\gamma_2^-)\Lambda^-]_{y,z}'\ , \qquad (31a)$$

$$[n^+(\gamma_1^++\gamma_2^+)\mathbf{\Lambda}^+-n^-(\gamma_1^--\gamma_2^-)\Lambda^-]_x=[n^+(\gamma_1^++\gamma_2^+)\mathbf{\Lambda}^+-n^-(\gamma_1^--\gamma_2^-)\Lambda^-]_x'\ . \qquad (31b)$$

In the next section I use these expressions to discuss the reflection and the refraction of plane waves.

V. PLANE WAVES IN HOMOGENEOUS ISOTROPIC MEDIA

I now consider the behavior of plane waves when there exists a discontinuity surface S. I discuss three possibilities: (i) material on both sides of S is achiral; (ii) the refracting medium is chiral; (iii) the reflecting medium is chiral.

A. Achiral material

For an achiral material one has $\alpha=\beta=0$ in (3) so that the solution of the system (11) is

$$n^{\pm}=\sqrt{\epsilon\mu}=n\ ,\quad p^{\pm}=\sqrt{\epsilon}=p\ ,\quad q^{\pm}=\sqrt{\mu}=q\ , \tag{32}$$

and according to (29)

$$\gamma_1^{\pm}=\tfrac{1}{2}\epsilon^{-1/2}\ ,\quad \gamma_2^{\pm}=\tfrac{1}{2}\mu^{-3/2}\ . \tag{32'}$$

Using (25) and (32) the expressions (26) simplify to

$$\Lambda_j^{\pm}(x_\nu,x_0)=\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}\int_0^{2\pi}e^{-ik_0n^{\pm}(x\cos\theta+z\sin\theta)}F_j^{\pm}(\theta)\psi^{\pm}(\theta)d\theta\ , \tag{33}$$

with

$$\mathbf{F}_1^{\pm}(\theta)=\mp i\sin\theta\ ,\quad \mathbf{F}_2^{\pm}(\theta)=1\ ,\quad \mathbf{F}_3^{\pm}(\theta)=\pm i\cos\theta\ , \tag{33'}$$

while the boundary conditions (31) reduce to

$$(\epsilon^{-1/2}+\mu^{-1/2})\Lambda_{y,z}^{+}+(\epsilon^{-1/2}-\mu^{-1/2})\Lambda_{y,z}^{-}=[\ ]_{y,z}'\ , \tag{34a}$$

$$n(\epsilon^{-1/2}+\mu^{-1/2})\Lambda_x^{+}-n(\epsilon^{-1/2}-\mu^{-1/2})\Lambda_x^{-}=n'[\ ]_x'\ . \tag{34b}$$

Let us now consider a plane wave with amplitude Q propagating in the first medium. When this wave reaches the interface S part of the wave is reflected and part is transmitted; that is, in mathematical terms,

$$\begin{aligned}\psi'^{\pm}(\theta)&=Q^{\pm}\delta(\theta-\theta_i)+R^{\pm}\delta(\theta-\theta_r)\ ,\\ \psi^{\pm}(\theta)&=T^{\pm}\delta(\theta-\theta_t)\ ,\end{aligned} \tag{35}$$

where $R^{\pm}$, $T^{\pm}$ are the reflected and refracted waves, δ is the Dirac distribution, and θ_i, θ_r, θ_t the angles of incidence, reflection and refraction, respectively. In agreement with (9) and (32) Q, R, T are defined by the relations

$$\begin{aligned}Q^{\pm}&=\sqrt{\epsilon'}Q_E\pm i\sqrt{\mu'}Q_H\ ,\\ R^{\pm}&=\sqrt{\epsilon'}R_E\pm i\sqrt{\mu'}R_H\ ,\\ T^{\pm}&=\sqrt{\epsilon}T_E\pm i\sqrt{\mu}T_H\ .\end{aligned} \tag{35'}$$

Recall the convention that letters with a prime concern the medium of the incident wave. Then substituting (35) into (33) gives

$$\Lambda_j'^{\pm}(x_\nu,x_0)=\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}\left\{Q^{\pm}\mathbf{F}_j^{\pm}(\theta_i)e^{ik_0n'(x\cos\theta_i+z\sin\theta_i)}+R^{\pm}F_j^{\pm}(\theta_r)e^{-ik_0n'(x\cos\theta_r+z\sin\theta_r)}\right\}\ , \tag{36a}$$

$$\Lambda_j^{\pm}(x_\nu,x_0)=\tfrac{1}{2}\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}T^{\pm}F_j^{\pm}(\theta_t)e^{-ik_0n(x\cos\theta_t+z\sin\theta_t)}\ . \tag{36b}$$

First note that at the boundary S the time variation of the secondary fields must be the same as that of the primary incident field which implies

$$n'\sin\theta_i=n'\sin\theta_r=n\sin\theta_t\ , \tag{37}$$

supplying the Descartes-Snell laws of reflection and refraction. Now, substituting (36) into the boundary conditions (34) gives the components R_E, R_H, T_E, T_H, in terms of Q_E, Q_H. Then using (37) one checks easily that the condition (34a) for the y component and the condition (34b) give the same relation

$$Q_E+R_E+i(Q_H+R_H)=T_E+iT_H\ , \tag{38a}$$

while the condition (34a) for the z component gives (since $\cos\theta_r=-\cos\theta_i$)

$$\cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}(Q_E-R_E)+i\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}(Q_H-R_H)=\cos\theta_t\left[\frac{\epsilon}{\mu}\right]^{1/2}T_E+i\cos\theta_t\left[\frac{\mu}{\epsilon}\right]^{1/2}T_H\ . \tag{38b}$$

From (38a) and (38b) a simple calculation yields the Fresnel relations

$$\begin{aligned}T_E&=\frac{2\sin\theta_t\cos\theta_i}{\sin\theta_t\cos\theta_i+(\mu'/\mu)\sin\theta_i\cos\theta_t}Q_E\ ,\quad R_E=\frac{\sin\theta_t\cos\theta_i-(\mu'/\mu)\sin\theta_i\cos\theta_E}{\sin\theta_t\cos\theta_i+(\mu'/\mu)\sin\theta_i\cos\theta_t}Q_E\\ T_H&=\frac{2\cos\theta_i\sqrt{\mu'/\epsilon'}}{\sqrt{\mu'/\epsilon'}\cos\theta_i+\sqrt{\mu/\epsilon}\cos\theta_t}Q_H\ ,\quad R_H=\frac{\sqrt{\mu'/\epsilon'}\cos\theta_i-\sqrt{\mu/\epsilon}\cos\theta_t}{\sqrt{\mu'/\epsilon'}\cos\theta_i+\sqrt{\mu/\epsilon}\cos\theta_t}Q_H\ .\end{aligned} \tag{39}$$

It is usual [6] to discuss the Fresnel relations in terms of the $E_\perp$, $E_\parallel$ components of Q, R, T, where $E_\perp$ and $E_\parallel$ denote, respectively, the electric field perpendicular and parallel to the plane of incidence. Since one has

$$(Q,R,T)_\perp=(Q,R,T)_E\ ,\quad (Q,R)_\parallel=\epsilon'^{-1/2}(Q,R)_H\ ,\quad T_H=\epsilon^{-1/2}T_H\ , \tag{40}$$

one checks easily that the expressions (39) are equivalent to the usual ones.

B. Chiral refractive medium

Now assume that the material is achiral in the first medium and chiral in the second one and recall that the incident wave is in the first medium for which I use the primed letters p', q', n', Λ'_j.

Taking into account (11′), take as the solution of the system (11) for a homogeneous chiral medium

$$p^\pm=\epsilon^{1/2}(1\pm i\alpha/n^\pm)^{1/2}\ ,\quad q^\pm=\mu^{1/2}(1\mp i\beta/n^\pm)^{1/2}\ , \tag{41}$$

$$n^\pm=\left[\left[\epsilon\mu-\frac{(\alpha+\beta)^2}{\mu}\right]^{1/2}\pm\frac{i}{2}(\beta-\alpha)\right]\ ,$$

and one has

$$p^\pm q^\pm=\epsilon\mu(n^\pm)^{-1}\ . \tag{42}$$

Substituting (41) into (29) and using (11′) a simple calculation gives

$$\begin{aligned}\gamma_1^\pm&=\frac{\epsilon^{-1/2}}{n^++n^-}\left[n^\pm(n^\pm\pm i\alpha)\right]^{1/2}\ ,\\ \gamma_2^\pm&=\frac{\mu^{-1/2}}{n^++n^-}\left[n^\pm(n^\pm\mp i\beta)\right]^{1/2}\ ,\end{aligned} \tag{43}$$

leading to

$$\begin{aligned}\gamma_1^++\gamma_2^+&=\frac{n^+}{n^++n^-}\left[\frac{p^+}{\epsilon}+\frac{q^+}{\mu}\right]\ ,\\ \gamma_1^--\gamma_2^-&=\frac{n^-}{n^++n^-}\left[\frac{p^-}{\epsilon}-\frac{q^-}{\mu}\right]\ .\end{aligned} \tag{43′}$$

In the achiral medium one has according to (32) and (32′)

$$p'^\pm=\sqrt{\epsilon'}=p'\ ,\quad q'^\pm=\sqrt{\mu'}=q'\ ,\quad n'^\pm=\sqrt{\epsilon'\mu'}=n'\ , \tag{44}$$

with

$$\gamma_1'^\pm=\tfrac{1}{2}\epsilon'^{-1/2}\ ,\quad \gamma_2'^\pm=\tfrac{1}{2}\mu'^{-1/2}\ . \tag{44′}$$

Then, using (43′) and (44′) the boundary conditions (31) become

$$\left[\frac{1}{\sqrt{\epsilon'}}+\frac{1}{\sqrt{\mu'}}\right]\Lambda_{y,z}'^{+}+\left[\frac{1}{\sqrt{\epsilon'}}-\frac{1}{\sqrt{\mu'}}\right]\Lambda_{y,z}'^{-}=\frac{n^+}{n^++n^-}\left[\frac{p^+}{\epsilon}+\frac{q^+}{\mu}\right]\Lambda_{y,z}^{+}+\frac{n^-}{n^++n^-}\left[\frac{p^-}{\epsilon}-\frac{q^-}{\mu}\right]\Lambda_{y,z}^{-}\ , \tag{45a}$$

$$n'\left[\frac{1}{\sqrt{\epsilon'}}+\frac{1}{\sqrt{\mu'}}\right]\Lambda_x'^{+}-n'\left[\frac{1}{\sqrt{\epsilon'}}-\frac{1}{\sqrt{\mu'}}\right]\Lambda_x'^{-}=\frac{(n^+)^2}{n^++n^-}\left[\frac{p^+}{\epsilon}+\frac{q^+}{\mu}\right]\Lambda_x^{+}-\frac{(n^-)^2}{n^++n^-}\left[\frac{p^-}{\epsilon}-\frac{q^-}{\mu}\right]\Lambda_x^{-}\ . \tag{45b}$$

Now, using (25), (41), and (44), the expressions (26) for the fields Λ'_j and Λ_j become

$$\Lambda_j'^\pm(x_v,x_0)=\frac{1}{2}\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}\left\{Q^\pm F_j^\pm(\theta_i)e^{-ik_0n'(x\cos\theta_i+z\sin\theta_i)}+R^\pm F_j^\pm(\theta_r)e^{-ik_0n'(x\cos\theta_r+z\sin\theta_r)}\right\}\ , \tag{46a}$$

$$\Lambda_j^\pm(x_v,x_0)=\frac{1}{2}\left[\frac{k_0}{\pi}\right]^{1/2}e^{ik_0x_0}T^\pm F_j^\pm(\theta_t^\pm)e^{-ik_0n^\pm(x\cos\theta_t^\pm+z\sin\theta_t^\pm)}\ , \tag{46b}$$

with

$$F_1^{\pm}(\theta)=\mp i\sin\theta\ ,\quad F_2^{\pm}(\theta)=1\ ,\quad F_3^{\pm}(\theta)=\pm i\cos\theta\ , \tag{47}$$

while in agreement with the relations (9), (41), (44) the fields Q, R, T are given by the expressions

$$\begin{aligned} Q^{\pm}&=\sqrt{\epsilon'}Q_E\pm i\sqrt{\mu'}Q_H\ ,\\ R^{\pm}&=\sqrt{\epsilon'}R_E\pm i\sqrt{\mu'}R_H\ ,\\ T^{\pm}&=p^{\pm}T_E\pm iq^{\pm}T_H\ . \end{aligned} \tag{48}$$

Let us first discuss the kinematic conditions for reflection and refraction,

$$n'\sin\theta_i=n'\sin\theta_r=n^{+}\sin\theta_t^{+}=n^{-}\sin\theta_t^{-}\ . \tag{49}$$

According to (41) n^+ and n^- are complex conjugate, consequently θ_t^+ and θ_t^- are also complex conjugate. Then, writing

$$n^{\pm}=n_1\pm in_2\ ,\quad \theta_t^{\pm}=\theta_1\pm i\theta_2\ , \tag{50}$$

I get from (49)

$$\begin{aligned} \sin\theta_1\cosh\theta_2&=\frac{n_1n'\sin\theta_i}{n_1^2+n_2^2}\ ,\\ \cos\theta_1\cosh\theta_2&=-\frac{n_2n'\sin\theta_i}{n_1^2+n_2^2}\ . \end{aligned} \tag{51}$$

These relations represent the generalized Descartes-Snell law for refraction of a plane wave in a chiral medium. One sees at once that Eqs. (51) have two solutions since they are invariant under the transformations.

Let us now discuss the boundary conditions (45). I first obtain from (11′) and (41) the relations

$$\frac{n^+}{n^++n^-}\frac{1}{\epsilon}(p^+)^2+\frac{n^-}{n^++n^-}\frac{1}{\epsilon}(p^-)^2=1=\frac{n^+}{n^++n^-}\frac{1}{\mu}(q^+)^2+\frac{n^-}{n^++n^-}\frac{1}{\mu}(q^-)^2\ , \tag{52a}$$

$$\frac{n^+}{n^++n^-}\frac{p^+q^+}{\epsilon}-\frac{n^-}{n^++n^-}\frac{p^-q^-}{\epsilon}=0=\frac{n^+}{n^++n^-}\frac{p^+q^+}{\mu}-\frac{n^-}{n^++n^-}\frac{p^-q^-}{\mu}\ . \tag{52b}$$

Then, I prove that the condition (45a) for the y component and the condition (45b) supply the same relation. Assuming that the kinematic conditions (49) are fulfilled I get from (45a) and (46)–(48) for the y component

$$\begin{aligned} &\left[\frac{1}{\sqrt{\epsilon'}}+\frac{1}{\sqrt{\mu'}}\right]\left\{\sqrt{\epsilon'}(Q_E+R_E)+i\sqrt{\mu'}(Q_H+R_H)\right\}+\left[\frac{1}{\epsilon'}-\frac{1}{\sqrt{\mu'}}\right]\left\{\sqrt{\epsilon'}(Q_E+R_E)-i\sqrt{\mu'}(Q_H+R_H)\right\}\\ &\qquad=\frac{n^+}{n^++n^-}\left[\frac{p^+}{\epsilon}+\frac{q^+}{\mu}\right]\left[p^+T_E+iq^+T_H\right]+\frac{n^-}{n^++n^-}\left[\frac{p^-}{\epsilon}-\frac{q^-}{\mu}\right](p^-T_E-iq^-T_H)\ , \end{aligned} \tag{53a}$$

while taking (49) into account I get from (45b)

$$\begin{aligned} &n'\sin\theta_i\left[\frac{1}{\sqrt{\epsilon'}}+\frac{1}{\sqrt{\mu'}}\right]\{\sqrt{\epsilon'}(Q_E+R_E)+i\sqrt{\mu'}(Q_H+R_H)\}+n'\sin\theta_i\left[\frac{1}{\sqrt{\epsilon'}}-\frac{1}{\sqrt{\mu'}}\right]\{\sqrt{\epsilon'}(Q_E+R_E)-i\sqrt{\mu'}(Q_H+R_H)\}\\ &\qquad=\frac{n'n^+\sin\theta_i}{n^++n^-}\left[\frac{p^+}{\epsilon}+\frac{q^+}{\mu}\right](p^+T_E+iq^+T_H)+\frac{n'n^-\sin\theta_i}{n^++n^-}\left[\frac{p^-}{\epsilon}-\frac{q^-}{\mu}\right](p^-T_E-iq^-T_H)\ . \end{aligned} \tag{53b}$$

Dividing (53b) by $n'\sin\theta_i$ supplies (53a). Now, taking into account (52a) and (52b) the relations (53) reduce to

$$Q_E+R_E+i(Q_H+R_H)=T_E+iT_H\ , \tag{54}$$

which is the same relation as (38a). Now consider (45a) for the z component. Still using (46)–(48) one has, since $\cos\theta_r=-\cos\theta_i$,

$$\begin{aligned} \cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}(Q_E-R_E)+i\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}(Q_H-R_H)&=\frac{n^+\cos\theta_t^+}{n^++n^-}\left[\frac{p^+}{\epsilon}+\frac{q^+}{\mu}\right](p^+T_E+iq^+T_H)\\ &\quad-\frac{n^-\cos\theta_t^-}{n^++n^-}\left[\frac{p^-}{\epsilon}-\frac{q^-}{\mu}\right](p^-T_E-iq^-T_H)\ . \end{aligned} \tag{55}$$

Then, taking into account (11′), (41), (50), (52a), and (52b) the relation (55) becomes

$$\begin{aligned} \cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}(Q_E-R_E)+i\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}(Q_H-R_H)&=\left\{\frac{\cos\theta_1\cosh\theta_2}{n^++n^-}[2\epsilon+i(\alpha+\beta)]+i\sin\theta_1\sinh\theta_2\right\}T_E\\ &\quad+\left\{\frac{\cos\theta_1\cosh\theta_2}{n^++n^-}[2\mu-i(\alpha+\beta)]+i\sin\theta_1\sinh\theta_2\right\}iT_H\ , \end{aligned} \tag{56}$$

then write

$$\cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}(Q_H-R_E)+i\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}(Q_H-R_H)=\left[\frac{\epsilon}{n_1}\cos\theta_1\cosh\theta_2+i\xi_E\right]T_E+\left[\frac{\mu}{n_1}\cos\theta_1\cosh\theta_2+i\xi_H\right]iT_H\ , \tag{57}$$

with

$$\xi_E=\sin\theta_1\sinh\theta_2+\frac{\alpha+\beta}{2n_1}\cos\theta_1\cosh\theta_2\ ,$$
$$\xi_H=\sin\theta_1\sinh\theta_2-\frac{\alpha+\beta}{2n_1}\cos\theta_1\cosh\theta_2\ . \tag{57'}$$

From (54) and (57) I get the relations

$$R_E=T_E-Q_E\ ,\quad R_H=T_H-Q_H\ , \tag{58a}$$

$$2\cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}Q_E=\left[\left[\frac{\epsilon'}{\mu'}\right]^{1/2}\cos\theta_i+\frac{\epsilon}{n_1}\cos\theta_1\cosh\theta_2\right]T_E-\xi_H T_H\equiv A_E T_E-\xi_H T_H\ , \tag{58b}$$

$$2\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}Q_H=\left[\left[\frac{\mu'}{\epsilon'}\right]^{1/2}\cos\theta_i+\frac{\mu}{n_1}\cos\theta_1\cosh\theta_2\right]T_H+\xi_E T_E\equiv\xi_E T_E+A_H T_H\ . \tag{58c}$$

From (58b) and (58c) I get

$$T_E=\frac{2\cos\theta_i}{A_E A_H+\xi_E\xi_H}\times\left[\left[\frac{\epsilon'}{\mu'}\right]^{1/2}A_H Q_E+\left[\frac{\mu'}{\epsilon'}\right]^{1/2}\xi_H Q_H\right]\ ,$$
$$T_H=\frac{2\cos\theta_i}{A_E A_H+\xi_E\xi_H}\times\left[\left[\frac{\mu'}{\epsilon'}\right]^{1/2}A_E Q_H-\left[\frac{\epsilon'}{\mu'}\right]^{1/2}\xi_E Q_E\right]\ , \tag{59a}$$

and substituting (59a) into (58a) leads to

$$R_E=\frac{1}{A_E A_H+\xi_E\xi_H}\left[B_E Q_E+2\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}\xi_H Q_H\right]\ ,$$
$$B_E=2\cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}A_H-A_E A_H-\xi_E\xi_H\ , \tag{59b}$$

$$R_H=\frac{1}{A_E A_H+\xi_E\xi_H}\left[B_H Q_H-2\cos\theta_i\left[\frac{\epsilon'}{\mu'}\right]^{1/2}\xi_E Q_E\right]\ ,$$
$$B_H=2\cos\theta_i\left[\frac{\mu'}{\epsilon'}\right]^{1/2}A_E-A_E A_H-\xi_E\xi_H\ . \tag{59c}$$

The expressions (59) are the Fresnel relations for the reflection and the refraction of a plane wave when the refractive medium is chiral. One checks easily that when $\alpha=\beta=0$ the expressions (59) reduce to the usual Fresnel relations (39).

C. Chiral reflective medium

Now consider the case where the incident wave propagates in a chiral medium. It is easy to transpose the results of the last section to this situation. First the kinematic conditions (49) become

$$n'^{\pm}\sin\theta_i=n'^{\pm}\sin\theta_r=n\sin\theta_t^{\pm}\ , \tag{60}$$

the angle of refraction $\theta_t^{\pm}$ is still complex and with $n'^{\pm}=n'_1\pm in'_2$, $\theta_t^{\pm}=\theta_1\pm i\theta_2$, we obtain

$$\sin\theta_1\cosh\theta_2=\frac{n'_1}{n}\sin\theta_i\ ,\quad \cos\theta_1\cosh\theta_2=\frac{n'_2}{n}\sin\theta_i\ , \tag{61}$$

which is the Descartes-Snell relation for refraction in a chiral reflective medium.

It is easy to check that the relation (54) is still valid while instead of (55) one has

$$\frac{\cos\theta_i}{2n'_1}[2\epsilon'+(\alpha'+\beta')](Q_E-R_E)+i\frac{\cos\theta_i}{2n'_1}[\epsilon\mu'-i(\alpha'+\beta')](Q_H-R_H)$$
$$=\frac{1}{2}\left[\frac{1}{\sqrt{\epsilon}}(\cos\theta_t^+-\cos\theta_t^-)+\frac{1}{\sqrt{\epsilon}}(\cos\theta_t^++\cos\theta_t^-)\right]\sqrt{\epsilon}T_E$$
$$+\frac{i}{2}\left[\frac{1}{\sqrt{\epsilon}}(\cos\theta_t^++\cos\theta_t^-)+\frac{1}{\sqrt{\mu}}(\cos\theta_t^+-\cos\theta_t^-)\right]\sqrt{\mu}T_H\ , \tag{62}$$

leading to

$$\frac{\epsilon'}{n'_1}\cos\theta_i(Q_E-R_E)+\frac{\alpha'+\beta'}{2n'_1}\cos\theta_i(Q_H-R_H)=\left[\frac{\epsilon}{\mu}\right]^{1/2}\cos\theta_1\cosh\theta_2T_E+\sin\theta_1\sinh\theta_2T_H\ ,$$
$$\frac{\mu'}{n'_1}\cos\theta_i(Q_H-R_H)+\frac{\alpha'+\beta'}{2n'_1}\cos\theta_i(Q_E-R_E)=\left[\frac{\mu}{\epsilon}\right]^{1/2}\cos\theta_1\cosh\theta_2T_H-\sin\theta_1\sinh\theta_2T_E\ . \tag{63}$$

Using (58a) to eliminate R_E and R_H, Eqs. (63) give

$$\frac{2\cos\theta_i}{n'_1}[\epsilon'Q_E+\tfrac{1}{2}(\alpha'+\beta')Q_H]=\left[\frac{\epsilon'}{n'_1}\cos\theta_i+\left[\frac{\epsilon}{\mu}\right]^{1/2}\cos\theta_1\cosh\theta_2\right]T_E+\left[\frac{\alpha'+\beta'}{2n'_1}\cos\theta_i+\sin\theta_1\sinh\theta_2\right]T_H\ ,$$
$$\frac{2\cos\theta_i}{n'_1}[\mu'Q_H+\tfrac{1}{2}(\alpha'+\beta')Q_E]=\left[\frac{\mu'}{n'_1}\cos\theta_i+\left[\frac{\mu}{\epsilon}\right]^{1/2}\cos\theta_1\cosh\theta_2\right]T_H+\left[\frac{\alpha'+\beta'}{2n'_1}\cos\theta_i-\sin\theta_1\sinh\theta_2\right]T_E\ . \tag{64}$$

The system (64) is easy to solve but it supplies intricate expressions. To simplify assume $\alpha'+\beta'=0$ and get

$$T_E=\frac{2\cos\theta_i}{n'_1}\left[\frac{\epsilon'K_HQ_E-\mu'\sin\theta_1\sinh\theta_2Q_H}{K_EK_H+\sin^2\theta_1\sinh^2\theta_2}\right],$$
$$T_H=\frac{2\cos\theta_i}{n'_1}\left[\frac{\mu'K_EQ_H+\epsilon'\sin\theta_1\sinh\theta_2Q_E}{K_EK_H+\sin^2\theta_1\sinh^2\theta_2}\right], \tag{65}$$

with

$$K_E=\frac{\epsilon'}{n'_1}\cos\theta_i+\left[\frac{\epsilon}{\mu}\right]^{1/2}\cos\theta_1\cosh\theta_2\ ,$$
$$K_H=\frac{\mu'}{n'_1}\cos\theta_i+\left[\frac{\mu}{\epsilon}\right]^{1/2}\cos\theta_1\cosh\theta_2\ . \tag{65'}$$

The expressions (58a) and (65) are the Fresnel relations for the reflection and the refraction of a plane wave when the reflective medium is chiral and when $\alpha'+\beta'=0$.

All three problems discussed in this section have been comprehensively handled by Lakthakia and Al [8,9].

VI. BREWSTER'S CONDITION

As recently discussed by Lakhtakia [10] the definition of the Brewster angle has changed along the years. Initially, the Brewster condtion implied that for an unpolarized light incident at the Brewster angle, the reflected light is plane polarized. In [10] a modern interpretation that is faithful to Brewster's original thoughts is given that is also more powerful than simply a zero-reflection condition.

Now let Q denote a wave packet with components Q_E, Q_H incident on an interface S. It generates a reflected wave R with components R_E, R_H and a refracted wave T with components T_E, T_H. In a general medium there exists a matrix relation between R, T, and Q,

$$\begin{bmatrix}R_E\\R_H\end{bmatrix}=\begin{vmatrix}R_{11}&R_{12}\\R_{21}&R_{22}\end{vmatrix}\begin{bmatrix}Q_E\\Q_H\end{bmatrix},$$
$$\begin{bmatrix}T_E\\T_H\end{bmatrix}=\begin{vmatrix}T_{11}&T_{12}\\T_{21}&T_{22}\end{vmatrix}\begin{bmatrix}Q_E\\Q_H\end{bmatrix}, \tag{66}$$

where R_{ij}, T_{ij} $(i,j,=1,2)$ are the coefficients of reflection and refraction, respectively.

Then, the Brewster condition is [11]

$$R_{11}R_{22}-R_{12}R_{21}=0\ . \tag{67}$$

In homogeneous isotropic achiral media one has $R_{12}=R_{21}=0$ so that (67) reduces to $R_{11}R_{22}=0$. Now according to (39) one checks easily that R_{11} cannot be zero while $R_{22}=0$ implies

$$n\mu'\cos\theta_i-n'\mu\cos\theta_t=0\ , \tag{68}$$

which is the usual relation [6] (taking into account our convention that the primed letters concern the medium of the incident wave).

Let us now consider the Brewster condition (67) for a chiral refractive medium. According to (59) one has

$$R_{11}=\frac{B_E}{A_EA_H+\xi_E\xi_H}\ ,\quad R_{12}=\frac{2\cos\theta_i\sqrt{\mu'/\epsilon'}\xi_H}{A_EA_H+\xi_E\xi_H}\ ,$$
$$R_{21}=\frac{-2\cos\theta_i\sqrt{\epsilon'/\mu'}\xi_E}{A_EA_H+\xi_E\xi_H}\ ,\quad R_{22}=\frac{B_H}{A_EA_H+\xi_E\xi_H}\ , \tag{69}$$

so that the condition (67) becomes

$$B_E B_H + 4\cos^2\theta_i \xi_E \xi_H = 0 \ . \tag{70}$$

To simplify the discussion of (70) assume $\alpha=\beta$, which implies

$$n_{2=0} \ , \quad n_1 = n = (\epsilon\mu - \alpha^2)^{1/2}, \quad \theta_2 = 0, \quad \theta_1 = \theta_t \ , \tag{71}$$

and according to (57′), (58b), and (58c) (further assume that $\mu'=1$)

$$\xi_E = -\xi_H = \frac{\alpha}{n}\cos\theta_t \ ,$$

$$A_E = \sqrt{\epsilon}\cos\theta_i + \frac{\epsilon}{n}\cos\theta_t \ , \tag{71′}$$

$$A_H = \frac{1}{\sqrt{\epsilon'}}\cos\theta_i + \frac{\mu}{n}\cos\theta_t \ ,$$

so that

$$A_E A_H + \xi_E \xi_H = \cos^2\theta_i + \cos^2\theta_t + \frac{1}{n}\cos\theta_i \cos\theta_t (\mu\sqrt{\epsilon'} + \epsilon/\sqrt{\epsilon'}) \ . \tag{72}$$

Then, from (59b) and (59c) I get

$$B_E = \cos^2\theta_i - \cos^2\theta_t + \frac{1}{n}\cos\theta_i\cos\theta_t \left[\mu\sqrt{\epsilon'} - \frac{\epsilon}{\sqrt{\epsilon'}}\right] \ ,$$
$$B_H = \cos^2\theta_i - \cos^2\theta_t - \frac{1}{n}\cos\theta_i\cos\theta_t \left[\mu\sqrt{\epsilon'} - \frac{\epsilon}{\sqrt{\epsilon'}}\right] \ . \tag{73}$$

Taking into account (71′) and (73) the condition (70) becomes

$$(\cos^2\theta_i - \cos^2\theta_t)^2 - \frac{1}{n^2}\cos^2\theta_i \times \cos^2\theta_t \left[\mu\sqrt{\epsilon'} - \frac{\epsilon}{\sqrt{\epsilon'}}\right]^2 - \frac{\alpha^2}{n^2}\cos^2\theta_t = 0 \ . \tag{74}$$

It is easy to check that for $a=0$ this condition reduces to (68). Equation (74) has the solutions $\cos\theta_i = \lambda\cos\theta_t$ which have to be consistent with the Descartes-Snell law while λ is solution of the equation

$$(\lambda^2-1)^2 - \lambda^2\frac{a^2}{n^2} - \frac{\alpha^2}{n^2} = 0 \ , \quad a = \mu\sqrt{\epsilon'} - \frac{\epsilon}{\sqrt{\epsilon'}} \ . \tag{75}$$

Using (58a) and (65) one should discuss in the same way the Brewster condition for a chiral reflective medium.

VII. CONCLUSION

As expected the complex formalism of electromagnetism covariant under the complex rotation group provides an useful tool to discuss electromagnetic fields in chiral media. For time-harmonic plane waves the Descartes-Snell laws, the Fresnel relations for reflection ad refraction, as well as the Brewster condition for plane-polarized reflected fields are obtained in a rather simple way. These results may be considered as justifying the use of the complex formalism. And they suggest to go on with more difficult problems first by expliciting the formalism for the general Maxwell equations and second by considering the case of anisotropic chiral media. It should also be interesting to consider the propagation of time-harmonic processes with nonlinear phases such as the focus-wave modes. Finally the physical interpretation of some of the mathematical expressions obtained here has to be discussed carefully.

The group SL(2,$\mathbb{C}$) of the 2×2 unimodular matrices is also isomorphic to L_0 leading to a spinor form of electromagnetism already investigated by Einstein and Maier [12] many years ago. Succinctly the electromagnetic spinor field is made of two traceless second-rank spinors ψ_r^s and φ_r^s ($r,s=1,2$, $\psi_1^1+\psi_2^2=0$, $\varphi_1^1+\varphi_2^2=0$) defined in an isotropic achiral medium by the relations

$$\psi_r^s = \sigma_{j,}^s(E_j + iH_j), \quad i=\sqrt{-1}, \quad j=1,2,3 \ ,$$
$$\varphi_r^s = \sigma_{j,r}^s(D_j + iB_j) \ , \tag{76}$$

where $\sigma_{j,r}^s$ are the components of the Pauli matrices. The Maxwell equations in the spinor form are

$$\partial_r^s \psi_s^t + \partial_{0,r}^s \varphi_s^t = 0 \ , \tag{77}$$

with

$$\partial_r^s = \sigma_{j,r}^s \partial_j, \quad \partial_{0,r}^s = \sigma_{0,r}^s \partial_{x_0} \ , \quad x_0 = \text{const} \ . \tag{77′}$$

Then, an interesting question is whether spinor formalism has a significant advantage on the three-dimensional complex formalism. To answer this question I first generalized (76) and (77) to an isotropic chiral medium and a spinor analysis of the problems discussed in Sec. V has been performed. Unfortunately the question of the better formalism is still open, awaiting the analysis of more complex situations. So I refrain from presenting this analysis since one obtains the same results as in Sec. V.

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